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On the stability of multiple helical vortices

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The classical problem of linear stability of a regular N-gon of point vortices to infinitesimal space displacements from an equilibrium of the vortex configuration is generalized to the one for N helical vortices (couple, triplet, etc., $N > 1$) for the first time. As a consequence of this consideration, the analytical form for the stability boundaries has been obtained. This solution allows an efficient analysis to be made of the existence of stable helical vortex arrays, which were repeatedly observed in practice.

Such a stability problem was earlier considered in theory, but only for the case of a plane polygonal array of N point vortices. As for helical vortices, owing to their complexity, intensive study has been mainly on the self-induced motion of the vortex.

The algebraic representation for the velocity of motion of the N helical vortex array was originally obtained as an additional intermediate result. The new formula allows accurate calculations to be made within the whole range of helical pitch variations and has a simpler form than the known asymptotic expressions.

Solution of these two classical problems of vortex dynamics has significance both for theoretic and applied mechanics, as well as for many other areas of natural science, where the rotor (vortex) concept is the basic one.

1. Introduction

The study of stability of some equilibrium vortex configurations is of special interest. For instance, the system of N longitudinal vortex structures, situated uniformly on a cylindrical surface with an identical angle displacement to the azimuth direction is one such problem. Owing to the complexity of the problem, the simpler problem of stability of stationary rotation of the point vortices (or rectilinear vortex filaments) with equal intensity Γ, situated in the vertices of a regular N-gon (figure 1) was first formulated by Thomson (Lord Kelvin) partially in view of his vortex atom theory. He drew attention to the similarity of the current problem and the problem of equilibrium for a system of uniform magnets floating in an external magnetic field (Thomson 1878). Obviously, in the state of equilibrium, the vortex polygonal array rotates, keeping its shape invariable with the angular velocity $\Omega = \Gamma(N-1)/4\pi a^2$, where a is the radius of a circumference rounding the vortices. Later on, J. J. Thomson (who discovered the electron) won the Adams prize for proof of the linear stability of a vortex polygonal array for $N < 7$ (Thomson 1883). The complete analysis of the stability for point vortices was carried out by Havelock (1931), who determined instability for $N > 7$ and considered $N = 7$ as a special case, for which linear analysis did not allow conclusions to be made on the stability of the system. He examined even more complex problems, taking into account, for example, global rotation with arbitrary tangentional velocity (not necessarily vortex free) and the presence of the second circular vortex array. The review of these and further work can be found in

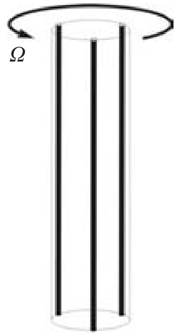

FIGURE 1. Scheme of a circular equilibrium array of N rectilinear vortices or polygonal array of N point vortices.

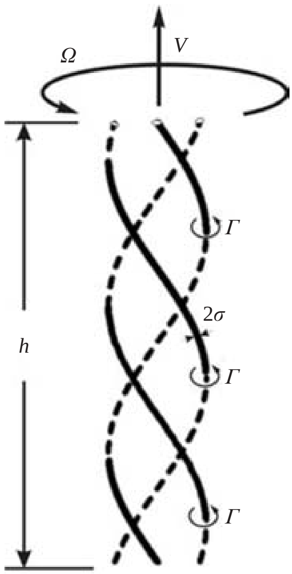

FIGURE 2. Scheme of a circular equilibrium array of N helical vortices.

Aref *et al.* (1988). Nonlinear stability analysis for the case of $N=7$ was carried out by Kurakin & Yudovich (2002).

These results obtained for particular cases, such as point or rectilinear vortices, essentially differ from practice, where vortices are usually turned into helical spirals (figure 2). Indeed, a stable couple of helical vortices was observed many times in various vortex flows (such as, tornado cores; after vortex breakdown over a delta wing, and in tubes, in vortex chambers of various geometries and for various uses; and as a model of the vortex couplings in turbulent flows). Triplets of helical vortices in the flows is a far less frequent case, while configuration of four vortex structures is an even more unstable phenomenon. In the preliminary experiments carried out in a rectangular vortex chamber, described by Alekseenko *et al.* (1999), we were able to observe four vortices within a very short time interval. Then, the structure was destroyed. In other words, real vortex configurations represent much more unstable objects than those predicted by point vortex models.

Another situation appears behind propellers, wind turbines, etc. when the vorticity is concentrated in N helical tip vortices and hub vortices of the negative sum circulation lying along the system axis. This $(N+1)$-vortex system is the simpler wake description proposed in the basic previous investigations (see, e.g. Joukowski 1912). Again, the problem of stability of the equilibrium circular configuration of the $(N+1)$-vortex system to infinitesimal space displacements was theoretically studied only for point vortices. In contrast to stability of N-point vortices without hub vortex for $N<7$, the $(N+1)$-point vortex system with hub vortex is absolutely unstable (see e.g. Morikawa & Swenson 1971). This result, obtained in the simplest plane case, is at variance with the numerous visualizations of wind-turbine and propeller wakes too where the helical tip vortices with small pitch usually exist with negligible changing in a long trail behind the turbine. The stability problem of the $(N+1)$ helical vortices has been studied by Okulov & Sorensen (2004) and the task is not considered here.

However, although stability of equilibrium configurations of N helical vortices is an important issue for vortex structure descriptions in tornado cores, after vortex breakdown, in vortex chambers, etc., the problem has not yet been considered analytically. So far, only part of the problem has been studied in detail – the problem of finding the self-induced velocity of motion for a single helical vortex in unbounded space; but, even this problem is complicated enough. The most significant achievements in this field, made by Kelvin (1880), Levy & Forsdyke (1928), Moore & Saffman (1972), Widnall (1972), etc., were reviewed by Ricca (1994). Two later papers were published by Kuibin & Okulov (1998) and Boersma & Wood (1999); and in addition to this Wood & Boersma (2001) considered motion of the N vortex system integrally. Finally, to note the contributions made by Russian scientists in solving this problem we must mention the paper published by Joukowski (1912). Notice also, that some of the investigations, concerning the helical vortices, are devoted to linear stability to infinitesimal sinusoidal disturbance of the helix form. Widnall (1972) describes the three main instability mode shapes as a 'short wave', a 'low wavenumber' and a 'mutual inductance' mode for small helical pitch. The non-local influence of the entire perturbed filament on the linear stability of a helix vortex is explored by Fukumoto & Miyazaki (1991) with the help of the cutoff method valid to the second order, which extends the first-order scheme developed by Widnall (1972). Gupta & Loewy (1974) and Bhagwat & Leishman (2000) extend the analysis for multiple helical vortices, though the vortex stability to sinusoidal perturbations of the helix form is not touched upon here.

The main goal of the current work is to find the conditions for linear stability to infinitesimal space displacements of equilibrium vortex configuration Λ, comprised of N identical slender helical vortices with circulation Γ (figure 2) and with the common pitch h or $l=h/2\pi$ (we introduce further dimensionless pitch $\tau=l/a$). The vortex axes of the system Λ are given by helical lines

$$X_n = (a\cos(\theta+n\delta);\quad a\sin(\theta+n\delta);\quad a\tau(\theta+n\delta)), \tag{1.1}$$

lying on a cylinder with radius a. The vortex central lines are azimuthally displaced by the angle $\delta=2\pi/N$. The vortices have circular cores with the same radius (referred to a) $\sigma \ll 1$. The vortex cores are a superposition of the helical vortex filaments which are collinear to central helical lines (1.1) and their vorticity is (instantaneously) uniformly distributed across the core cross-section (see e.g. Ricca 1994). Besides, the vortex system Λ under consideration should meet additional conditions: $2\sigma<\tau$ and $N<\pi/\sigma$. Such a non-perturbed vortex system rotates and moves uniformly along the axis of a cylinder. Incidentally, determination of the moving velocity of the circular equilibrium array of helical vortices still cannot be considered as a solved problem. Indeed, Wood & Boersma (2001) have obtained the integral representation for binormal velocity of the system motion; however, they wrote down a closed analytical form of the solution only for velocity asymptotic expansions at large and small values of dimensionless pitch, and only for some vortices in the system ($N=2$, 3 and 4).

In contrast to rectilinear filaments (or point vortices) where the induced velocity has a simple solution in the form of a pole, the Biot-Savart law for helical filaments cannot be integrated in a closed form. Only asymptotic solutions can be written analytically, but they cannot provide high accuracy within the whole range of variation of helical vortex pitch, when calculating the velocity field necessary to solve the stability problem (see for instance, Ricca 1994; Boersma & Wood 1999). Another form of the solution based on infinite series from products of modified cylindrical functions (Kapteyn series) was found by Hardin (1982) for a helical vortex filament in infinite fluid, and generalized by Okulov (1993, 1995) for the filament in an cylindrical tube, coaxial to the filament axis. This presentation of the solution does not simplify the calculation technique because it is a divergent series on the vortex filaments. Therefore, to study the problem we have to start by developing a simple procedure, accurate enough for calculating the velocity field induced by helical vortex filaments. To simplify calculations, we limit our study by considering only infinite fluid.

2. Velocity field induced by a circular array of helical vortex filaments

According to Hardin's (1982) solution in cylindrical coordinates (r, θ, z), the components of fluid velocity induced by the N vortices outside the vortex cores are given as,

$$u_r(r,a,\chi) = \frac{\Gamma a}{\pi l^2}\,\mathrm{Im}\sum_{n=1}^{N}\left\{\begin{matrix} H_1^{1,1}(r/l, a/l, \chi - 2\pi n/N) \\ H_1^{1,1}(a/l, r/l, \chi - 2\pi n/N)\end{matrix}\right\}, \tag{2.1a}$$

$$u_\theta(r,a,\chi) = \frac{\Gamma}{2\pi r}\left\{\begin{matrix} 0 \\ N\end{matrix}\right\} + \frac{\Gamma a}{\pi r l}\,\mathrm{Re}\sum_{n=1}^{N}\left\{\begin{matrix} H_1^{0,1}(r/l, a/l, \chi - 2\pi n/N) \\ H_1^{1,0}(a/l, r/l, \chi - 2\pi n/N)\end{matrix}\right\}, \tag{2.1b}$$

$$u_z(r,a,\chi) = \frac{\Gamma}{2\pi l}\left\{\begin{matrix} N \\ 0\end{matrix}\right\} - \frac{\Gamma a}{\pi l^2}\,\mathrm{Re}\sum_{n=1}^{N}\left\{\begin{matrix} H_1^{0,1}(r/l, a/l, \chi - 2\pi n/N) \\ H_1^{1,0}(a/l, r/l, \chi - 2\pi n/N)\end{matrix}\right\}, \tag{2.1c}$$

and the respective streamfunction is written in the following form,

$$\psi = \frac{\Gamma N}{4\pi}\left\{\begin{matrix} r^2/l^2 - \ln a^2 \\ a^2/l^2 - \ln r^2\end{matrix}\right\} - \frac{\Gamma a r}{\pi l^2}\,\mathrm{Re}\sum_{n=1}^{N}\left\{\begin{matrix} H_0^{1,1}(r/l, a/l, \chi - 2\pi n/N) \\ H_0^{1,1}(a/l, r/l, \chi - 2\pi n/N)\end{matrix}\right\}, \tag{2.2}$$

where $\chi = \theta - z/l$, and the upper expression in braces corresponds to the case of $r < a$, and the lower to $r > a$. The sum in these formulae all involve Kapteyn series and is written as follows:

$$H_M^{I,J}(x,y,\chi) = \sum_{n=1}^{\infty} m^M I_m^{\langle I\rangle}(mx) K_m^{\langle J\rangle}(my)\exp(\mathrm{i}m\chi), \tag{2.3}$$

where $x \leqslant y$, $I_m^{\langle 0\rangle}(mx)$, $K_m^{\langle 0\rangle}(my)$ are the modified Bessel functions and $I_m^{\langle 1\rangle}(mx)$, $K_m^{\langle 1\rangle}(my)$ are their respective derivatives.

Note that formulae (2.1)–(2.3) depend only upon two helical variables r and χ. This means that the solution outside of Λ belongs to a class of flows with helical symmetry. It follows immediately from (2.1), that the tangential velocity to the helical lines (1.1) remains constant

$$u_\tau \equiv u_z + r u_\theta / l = \Gamma N / 2\pi l \equiv \text{const.} \tag{2.4}$$

Similar to the translation fluid motion with constant velocity in the whole space, the fluid flow with helical symmetry satisfying condition (2.4) can be tentatively called a flow with 'translation motion' along the helical lines. Analysing (2.1), we can conclude that the considered class of flows can be regarded as a 'translation motion' only conventionally. Indeed, the velocity components u_r, u_θ and u_z can take on arbitrary values in space, in fulfilling the relation (2.4), which bounds just axial and circumferential velocity components. In an extreme case, when $l \to \infty$, and helical lines become straight, the flow towards the z-axis vanishes; the vortex filament becomes rectilinear and induces only circular motion around its own axis, while the axial component of the velocity becomes equal to zero. For the velocity component $u_\chi = u_\theta - r u_z / l$, which is orthogonal to u_r and to u_τ , we obtain from (2.1):

$$u_\chi(r, a, \chi) = \frac{\Gamma N}{2\pi} \begin{Bmatrix} -r/l^2 \\ 1/r \end{Bmatrix} + \frac{\Gamma a}{\pi l^2} \left(\frac{l}{r} + \frac{r}{l} \right) \operatorname{Re} \sum_{n=1}^{N} \begin{Bmatrix} H_1^{0,1}(r/l, a/l, \chi - 2\pi n/N) \\ H_1^{1,0}(a/l, r/l, \chi - 2\pi n/N) \end{Bmatrix}. \tag{2.5}$$

We analyse in detail the flows with helical symmetry to prove a validation of the relation (2.4) for all points of the flow field including the vortex cores of Λ. Using helical variables r and χ, the problem can be reduced formally to two-dimensional continuity and Euler equations:

$$\frac{\partial (r u_r)}{\partial r} = -\frac{\partial u_\chi}{\partial \chi}, \tag{2.6a}$$

$$\frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} + u_\chi \frac{\partial u_r}{r \partial \chi} - \frac{1}{r} \frac{l^4}{(r^2 + l^2)^2} \left(\frac{r}{l} u_\tau + u_\chi \right)^2 = -\frac{1}{\rho} \frac{\partial p}{\partial r}, \tag{2.6b}$$

$$\frac{\partial}{\partial t} u_\chi + u_r \frac{\partial u_\chi}{\partial r} + u_\chi \frac{\partial u_\chi}{r \partial \chi} + \frac{u_\tau u_r}{r} = -\frac{1}{\rho} \frac{l^2 + r^2}{l^2} \frac{\partial p}{r \partial \chi}, \tag{2.6c}$$

$$\frac{\partial}{\partial t} u_\tau + u_r \frac{\partial u_\tau}{\partial r} + u_\chi \frac{\partial u_\tau}{r \partial \chi} \equiv 0, \tag{2.6d}$$

with velocity components $u_r, u_\tau = u_z + r u_\theta / l$ and $u_\chi = u_\theta - r u_z / l$, which correspond to those in cylindrical coordinates and with an analogue of the streamfunction by the following equations:

$$u_r = \frac{1}{r} \frac{\partial \psi}{\partial \chi}, \quad u_\chi = -\frac{\partial \psi}{\partial r}, \tag{2.7a}$$

$$u_\theta = \frac{l^2}{r^2 + l^2} \left(\frac{r}{l} u_\tau + u_\chi \right) = \frac{l^2}{r^2 + l^2} \left(\frac{r}{l} u_\tau - \frac{\partial \psi}{\partial r} \right), \tag{2.7b}$$

$$u_z = \frac{l^2}{r^2 + l^2} \left(u_\tau - \frac{r}{l} u_\chi \right) = \frac{l^2}{r^2 + l^2} \left(u_\tau + \frac{r}{l} \frac{\partial \psi}{\partial r} \right). \tag{2.7c}$$

Equation (2.6*d*) defines a simple class of solutions with $u_\tau = \text{const}$ (Okulov 1993; 1995). The vorticity vector in helical coordinates for the given class of flows can be written as

$$\omega_r = \frac{1}{r} \frac{\partial u_\tau}{\partial \chi} \equiv 0, \tag{2.8a}$$

$$\omega_\theta = -\frac{1}{l}\frac{\partial u_r}{\partial \chi} - \frac{\partial u_z}{\partial r} = -\frac{r}{l}\left[\frac{1}{r^2}\frac{\partial^2 \psi}{\partial \chi^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{rl^2}{r^2+l^2}\frac{\partial \psi}{\partial r}\right) - \frac{2u_\tau}{l}\left(\frac{l^2}{r^2+l^2}\right)^2\right], \quad (2.8b)$$

$$\omega_z = -\frac{1}{r}\frac{\partial u_r}{\partial \chi} + \frac{1}{r}\frac{\partial (ru_\theta)}{\partial r} = -\left[\frac{1}{r^2}\frac{\partial^2 \psi}{\partial \chi^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{rl^2}{r^2+l^2}\frac{\partial \psi}{\partial r}\right) - \frac{2u_\tau}{l}\left(\frac{l^2}{r^2+l^2}\right)^2\right]. \quad (2.8c)$$

Condition $u_\tau = \text{const}$ for all points of the flow including the vortex cores of Λ shows that a radial component of the vorticity field in (2.8*a*) is equal to zero, while the analysis of (2.8*b*) and (2.8*c*) allow us to get the correlation between the axial and circumferential components of the vorticity vector

$$\omega_r = 0, \quad \omega_\theta/\omega_z = r/l \quad \text{or} \quad \omega_\theta = r\omega_z/l. \quad (2.9a\text{–}c)$$

From (2.8*b*) and (2.8*c*) we find the equation to determine a streamfunction ψ based on a given distribution of axial component of vorticity ω_z

$$\frac{\partial^2 \psi}{\partial r^2} + \left(1 - \frac{2r^2}{r^2+l^2}\right)\frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{l^2+r^2}{r^2 l^2}\frac{\partial^2 \psi}{\partial \chi^2} = \frac{2u_\tau}{l}\left(\frac{l^2}{r^2+l^2}\right)^2 - \frac{\omega_z(r^2+l^2)}{l^2}, \quad (2.10)$$

which was used to solve the problem of a helical vortex filament in a cylindrical tube (Okulov 1993, 1995).

Equations (2.8) mean that for the given class of flows with $u_\tau = \text{const}$, the vorticity vector is always directed tangentially to the helical lines. The opposite statement is also true (Okulov 2002): if the vorticity field is collinear to helical lines (1.1), then condition (2.4) holds true for all points of the flow field including the vortex cores of Λ. Indeed, if we apply condition (2.9) to the vorticity field in the flows with helical symmetry, then after integrating (2.9*a*) and taking into consideration the determination of ω_r, we will find $u_\tau = f(r)$. According to (2.9*b*) the difference $\omega_\theta - r\omega_z/l$ is zero. Taking into account (2.7) and (2.8), we obtain $f'(r) = 0$, i.e. $u_\tau = \text{const}$. Thus, fulfilment of the condition (2.4), applied to the whole of the velocity field, is equivalent to (2.9), i.e. to the requirement of collinearity of the vorticity field with tangents directed to the helical lines. In this case, any distribution of the vorticity, satisfying (2.9), including the considered vortex configuration Λ of identical helical vortices, is described by linear equation (2.10). Therefore, the superposition principle used in (2.1) and (2.2) to determine the total induced velocity field outside the vortex cores is justified for the whole of the flow field. So now it is enough to develop a calculation technique for kinematic characteristics of the flow induced by a single filament.

3. The evaluation of the Kapteyn series

Note that $H_M^{I,J}(x, y, \chi)$ series converge slowly, especially when approaching the singularity ($r \to a$ and $\chi \to 0$), which complicates the calculation of the kinematic characteristics of the flow (2.1)–(2.2). Let us consider some methods used to calculate series (2.3) in the problem of self-induced motion of a helical vortex filament (see Ricca 1994; Boersma & Wood 1999). The binormal component of the velocity u_b in dimensionless form (i.e. dividing by $\Gamma/4\pi a$) was calculated at the points situated on a helical surface $\theta - z/l = 0$ at a small distance from the vortex filament $\sigma = \varepsilon R$ (where $R = (a^2 + l^2)/a = a(1+\tau^2)$ and $\tau = l/a$). Following Ricca's (1994) notation in the interior point, $r = a - \varepsilon R$ for $\widehat{v} = u_b R/a - 2/\varepsilon = u_b(1+\tau^2) - 2/\varepsilon$, we have:

$$\widehat{v}_{\text{int}} = 2\frac{(1+\tau^2)^{1/2}}{\tau} - \frac{4(1-\varepsilon)(1+\tau^2)^{3/2}}{\tau^2[1-\varepsilon(1+\tau^2)]}\operatorname{Re}\left\{H_1^{0,1}\left(\frac{1-\varepsilon(1+\tau^2)}{\tau}, \frac{1}{\tau}\right)\right\} - \frac{2}{\varepsilon}, \quad (3.1)$$

and in the exterior point, $r=a+\varepsilon R$ for $\widehat{v}=u_b R/a+2/\varepsilon=u_b(1+\tau^2)+2/\varepsilon$,

$$\widehat{v}_{\mathrm{ext}}=-\frac{2\tau(1+\tau^2)^{1/2}}{1+\varepsilon(1+\tau^2)}-\frac{4(1+\varepsilon)(1+\tau^2)^{3/2}}{\tau^2[1+\varepsilon(1+\tau^2)]}\,\mathrm{Re}\left\{H_1^{1,0}\left(\frac{1}{\tau},\frac{1+\varepsilon(1+\tau^2)}{\tau}\right)\right\}+\frac{2}{\varepsilon}. \tag{3.2}$$

Ricca attempted to calculate (3.1) and (3.2) by improving the accuracy of the cylindrical functions, contained in the $H_1^{0,1}$ and $H_1^{1,0}$ series. The calculation results for three ranges of helical pitch variations $\tau=\{0,5\div 4\}$, $\{5\div 12\}$ and $\{29\div 37\}$ at the values of $\varepsilon=0.1$, 0.01 and 0.001, respectively, were presented graphically in his figure 6 (Ricca 1994). However, the necessity of terminating the series in these calculations means that he cannot avoid the problem of error accumulation when calculation points approach singular points ($\varepsilon\to 0$ and $\tau\to 0$ or ∞). In order to overcome this difficulty, Boersma & Wood (1999) removed the singularities from the series (2.3) by simply adding and subtracting the pole and logarithm in spatial variables for two particular cases of the series (2.3) used in (3.1) and (3.2),

$$H_1^{0,1}\left(\frac{1-\varepsilon(1+\tau^2)}{\tau},\frac{1}{\tau},0\right)=\frac{1}{4}\frac{\tau^2}{(1+\tau^2)^{3/2}}\left(-\frac{2}{\varepsilon}+\ln(\varepsilon)+\ln\left(\frac{\sqrt{1+\tau^2}}{2}\right)\right)$$
$$+\frac{\tau}{2}-\frac{\tau^2}{4}W(\tau)+o(1), \tag{3.3}$$

$$H_1^{1,0}\left(\frac{1}{\tau},\frac{1+\varepsilon(1+\tau^2)}{\tau},0\right)=\frac{1}{4}\frac{\tau^2}{(1+\tau^2)^{3/2}}\left(\frac{2}{\varepsilon}+\ln(\varepsilon)+\ln\left(\frac{\sqrt{1+\tau^2}}{2}\right)\right)$$
$$-\frac{\tau^2}{4}W(\tau)+o(1), \tag{3.4}$$

where $o(1)$ is Landau's symbol for an expression that tends to 0 when $\varepsilon\to 0$. Moreover, Boersma & Wood (1999) employed an integral form found in Boersma & Yakubovich (1998) for the difference W between series of the Kapteyn type in the form of (2.3) and terms with singularities from (3.3)–(3.4):

$$W(\tau)=\int_0^\infty\left\{\frac{\sin^2 t}{[\tau^2t^2+\sin^2 t]^{3/2}}-\frac{1}{[\tau^2+1]^{3/2}}\frac{H(1/2-t)}{t}\right\}\mathrm{d}t, \tag{3.5}$$

where $H(\cdot)$ denotes the unit step function. The integral remainder (3.5) describes the main effect of vortex torsion and is regular, but cannot be integrated in a closed form just as the Biot-Savart law for the helical vortex filament. Therefore, it was numerically calculated to six significant figures and given in table 1 of Boersma & Wood (1999) for twenty-one values of the pitch τ. By means of (3.5), Boersma & Wood (1999) specified the known asymptotic forms for large and small values of dimensionless pitch τ (Ricca 1994; Kuibin & Okulov 1998). Unfortunately, the asymptotic forms are inaccurate in the practically important variation range of τ from 0.5 to 3, even after this improvement.

Another example of computation of series (2.3) concerns calculation of a binormal velocity component on the cylinder containing the helical vortex filaments. This calculation is associated with the determination of the moving velocity of an array of N identical helical vortex filaments located on a cylinder with identical angle of displacement $\delta=2\pi/N$. The dimensionless binormal velocity component in Wood & Boersma (2001) was represented as

$$u_b=-\frac{4\pi a}{\Gamma}\frac{\tau}{(1+\tau^2)^{1/2}}u_\chi=(1+\tau^2)^{1/2}W(\alpha,\tau)-\frac{2\tau}{(1+\tau^2)^{1/2}}, \tag{3.6}$$

where α is the angular coordinate from a vortex filament to a reference point, while integral

$$W(\alpha,\tau)=\int_0^{\infty}\left\{\frac{\sin^2(t-\alpha/2)}{[\tau^2t^2+\sin^2(t-\alpha/2)]^{3/2}}\right\}\mathrm{d}t,\qquad 0<\alpha<2\pi, \tag{3.7}$$

just as the integral (3.5), could not be integrated in a closed form. Note that the integral (3.7) was calculated for non-zero values of $\alpha=k\delta$ ($k=1,\ldots,N-1$), since for $\alpha=0$ this integral diverges. Moreover, Wood & Boersma (2001) ascertained the relationship with the integral (3.5), by subtracting singularity from (3.7) within the limit at $\alpha\to 0$, which can be considered as a finite part of (3.7) when $\alpha\to 0$. However, for testing other computation methods, it is more important to note that the values (3.7) were calculated to six significant figures and given in table 1 (Wood & Boersma 2001) at $\alpha=\pi/2, 2\pi/3, \pi$ for twenty-one values of τ, the same way as in Boersma & Wood (1999).

Let us consider another approach as a more effective evaluation procedure of series of the Kapteyn type written in form of (2.3). This approach to determine the velocity and streamfunction, gives the series (2.3) with direct extraction of singularities in distorted spatial variables, expressly taking into consideration the vortex torsion (Okulov 1993, 1995). In contrast to the above described methods, this approach allows us to calculate the flow characteristics (2.1)–(2.2) for any point using elementary functions. Further on, we will generalize this method for series (2.3) of any of their types (i.e. I and $J=0$ or 1), appearing in determinations of streamfunction ($M=0$), velocity field ($M=1$), and its spatial derivatives ($M=2$), determined in Appendix A. To this end, in series (2.3) we formally substitute cylindrical functions by their uniform expansions at high orders (see Appendix B). The resulting series can be reduced to the closed forms (see Appendix C), so that the main part of the series (2.3) $S_M^{I,J}$ takes the form:

$$\begin{aligned}S_M^{I,J}=\lambda^{I,J}\Bigg[&b_{M,0}^{I,J}\frac{\mathrm{e}^{\xi+\mathrm{i}\chi}}{(\mathrm{e}^{\xi}-\mathrm{e}^{\mathrm{i}\chi})^2}+b_{M,1}^{I,J}\frac{\mathrm{e}^{\mathrm{i}\chi}}{\mathrm{e}^{\xi}-\mathrm{e}^{\mathrm{i}\chi}}+b_{M,2}^{I,J}\ln(1-\mathrm{e}^{-\xi+\mathrm{i}\chi})\\&+b_{M,3}^{I,J}\mathrm{Li}_2(\mathrm{e}^{-\xi+\mathrm{i}\chi})+b_{M,4}^{I,J}\mathrm{Li}_3(\mathrm{e}^{-\xi+\mathrm{i}\chi})+b_{M,5}^{I,J}\mathrm{Li}_4(\mathrm{e}^{-\xi+\mathrm{i}\chi})\Bigg],\end{aligned} \tag{3.8}$$

$$\mathrm{e}^{\xi}=\frac{x}{y}\frac{\exp(\sqrt{1+x^2})(1+\sqrt{1+y^2})}{\exp(\sqrt{1+y^2})(1+\sqrt{1+x^2})},\qquad \lambda^{I,J}=\tfrac{1}{2}\frac{(\sqrt{1+x^2})^{I-1/2}(\sqrt{1+y^2})^{J-1/2}}{x^I(-y)^J},$$

$$b^{I,J}=\begin{bmatrix}0&0&1&\alpha^{I,J}&\beta^{I,J}&\gamma^{I,J}\\0&1&\alpha^{I,J}&\beta^{I,J}&\gamma^{I,J}&0\\1&\alpha^{I,J}&\beta^{I,J}&\gamma^{I,J}&0&0\end{bmatrix},$$

where $\mathrm{Li}_k(z)=\sum_{m=1}^{\infty}(z^m/m^k)$, $|z|<1$ are the polylogarithms, $\mathrm{Li}_0(z)=\ln(1-z)$; $\alpha^{I,J}$, $\beta^{I,J}$, $\gamma^{I,J}$ are the polynomials of 1–3 degree, obtained from ϑ_i and υ_i functions by multiplication of the uniform expansion from Appendix B. The possibility of calculating the kinematic characteristics of (2.1)–(2.2) through replacing series $H_M^{I,J}$ by their main part $S_M^{I,J}$ was evaluated comparing with velocity values, calculated based on W magnitudes, taken from table 1 of Wood & Boersma (2001). As a result, it was determined that the maximum deviation takes place for τ values from 0.5 to 3, where asymptotics are not valid. This deviation is not significant – just around 2%, which is acceptable for calculations of the velocity field for any points. Taking into

τ	$W(\tau)$ (3.5) by WB	$W(\tau)$ (4.9) present data	$W(\tau)$ Difference (%)	$W(\pi/2,\tau)$ (3.7) by WB	$W(\pi/2,\tau)$ (4.5) present data	$W(\pi/2,\tau)$ Difference (%)	$W(3\pi/2,\tau)$ (3.7) by WB	$W(3\pi/2,\tau)$ (4.5) present data	$W(3\pi/2,\tau)$ Difference (%)	$W(\pi,\tau)$ (3.7) by WB	$W(\pi,\tau)$ (4.5) present data	$W(\pi,\tau)$ Difference (%)
0.01	95.7022	95.7022	0.0	99.6535	99.6535	0.0	99.4508	99.4508	0.0	99.3069	99.3069	0.0
0.05	17.3173	17.3173	0.0	19.6546	19.6546	0.0	19.4523	19.4523	0.0	19.3086	19.3086	0.0
0.1	8.01822	8.01822	0.0	9.65819	9.65819	0.0	9.45703	9.45703	0.0	9.31407	9.31407	0.0
0.2	3.70710	3.70711	−0.0005	4.67226	4.67226	0.0	4.47767	4.47767	0.0	4.33908	4.33908	0.0
0.3	2.39240	2.39257	−0.0173	3.02815	3.02801	0.0143	2.84996	2.84990	−0.0057	2.72379	2.72392	0.0134
0.4	1.74543	1.74601	−0.0579	2.22381	2.22333	0.0477	2.06975	2.06956	−0.0192	1.96213	1.96257	0.0437
0.5	1.34138	1.34214	−0.0759	1.75584	1.75524	0.0602	1.62671	1.62649	−0.0224	1.53777	1.53832	0.0551
0.6	1.05695	1.05730	−0.035	1.45425	1.45400	0.0250	1.34689	1.34681	−0.0081	1.27372	1.27393	0.0212
0.7	0.844909	0.844428	0.0481	1.24544	1.24587	−0.0425	1.15576	1.15594	0.0184	1.09502	1.09463	−0.0395
0.8	0.682350	0.681017	0.1333	1.09273	1.09382	−0.1092	1.01707	1.01749	0.0429	0.965957	0.964960	−0.0997
0.9	0.555822	0.553923	0.1899	0.976115	0.977631	−0.1516	0.911533	0.912114	0.0581	0.867901	0.866524	−0.1377
1	0.456367	0.454283	0.2084	0.883919	0.885561	−0.1642	0.828167	0.828789	0.0622	0.790427	0.788942	−0.1485
2	0.092836	0.093583	−0.0746	0.468166	0.467590	0.0576	0.448986	0.448773	−0.0213	0.435385	0.435903	0.0518
3	0.030892	0.031366	−0.0475	0.321010	0.320630	0.0380	0.311638	0.311492	−0.0146	0.304712	0.305057	0.0345
4	0.013608	0.013752	−0.0144	0.244085	0.243964	0.0121	0.238716	0.238667	−0.0049	0.234623	0.234735	0.0112
5	0.007112	0.007131	−0.0019	0.196740	0.196720	0.0020	0.193348	0.193338	−0.0010	0.190702	0.190721	0.0019
6	0.004162	0.004142	0.0020	0.164690	0.164704	−0.0014	0.162397	0.162401	0.0004	0.160576	0.160564	−0.0012
7	0.002639	0.002609	0.0030	0.141572	0.141595	−0.0023	0.139943	0.139951	0.0008	0.138630	0.138610	−0.0020
8	0.001776	0.001746	0.0030	0.124120	0.124143	−0.0023	0.122917	0.122925	0.0008	0.121935	0.121914	−0.0021
9	0.001251	0.001225	0.0027	0.110482	0.110503	−0.0021	0.109567	0.109575	0.0008	0.108812	0.108793	−0.0019
10	0.000914	0.000891	0.0023	0.099536	0.099554	−0.0018	0.098821	0.098828	0.0007	0.098227	0.098210	−0.0017

TABLE 1. Comparison of numerical evolution of $W(\tau)$ and $W(\alpha,\tau)$ for $\alpha=\pi/2, 3\pi/2, \pi$ and values of τ indicated. The present data are based on (4.5) and (4.9). The results for (3.5) and (3.7) were taken from Wood & Boersma (2001) – WB, table 1.

consideration a few terms of the regular remainder we reduce even this error:

$$R_M^{I,J}(x,y,\chi) = H_M^{I,J}(x,y,\chi) - S_M^{I,J}(x,y,\chi),$$

where

$$R_M^{I,J} = \sum_{m=1}^{\infty}\left[m^M I_m^{\langle I\rangle}(mx)K_m^{\langle J\rangle}(my) - m^M\lambda^{I,J}(\mathrm{e}^{\xi})^m\left(1+\frac{\alpha^{I,J}}{m}+\frac{\beta^{I,J}}{m^2}+\frac{\gamma^{I,J}}{m^3}\right)\right]\mathrm{e}^{\mathrm{i}m\chi}. \quad (3.9)$$

Figure 3 illustrates the analysis of the influence of the first three terms of the remainder $R_M^{I,J}$,

$$\Delta R_1(r,\tau,\chi) = \left\{\begin{matrix}\left[I_1^{\langle 0\rangle}\left(\frac{r}{\tau}\right)K_1^{\langle 1\rangle}\left(\frac{1}{\tau}\right) - \lambda^{\langle 0,1\rangle}\,\mathrm{e}^{\xi}(1+\alpha^{0,1}+\beta^{0,1}+\gamma^{0,1})\right]\mathrm{e}^{\mathrm{i}\chi}\Big/ S_1^{0,1}\left(\frac{r}{\tau},\frac{1}{\tau},\chi\right)\\ \left[I_1^{\langle 1\rangle}\left(\frac{1}{\tau}\right)K_1^{\langle 0\rangle}\left(\frac{r}{\tau}\right) - \lambda^{\langle 1,0\rangle}\,\mathrm{e}^{\xi}(1+\alpha^{1,0}+\beta^{1,0}+\gamma^{1,0})\right]\mathrm{e}^{\mathrm{i}\chi}\Big/ S_1^{1,0}\left(\frac{1}{\tau},\frac{r}{\tau},\chi\right)\end{matrix}\right\},$$

$$\Delta R_2(r,\tau,\chi) = 2\left\{\begin{matrix}\left[I_2^{\langle 0\rangle}\left(\frac{2r}{\tau}\right)K_2^{\langle 1\rangle}\left(\frac{2}{\tau}\right) - \lambda^{\langle 0,1\rangle}\,\mathrm{e}^{2\xi}\left(1+\frac{\alpha^{0,1}}{2}+\frac{\beta^{0,1}}{4}+\frac{\gamma^{0,1}}{8}\right)\right]\mathrm{e}^{2\mathrm{i}\chi}\Big/ S_1^{0,1}\left(\frac{r}{\tau},\frac{1}{\tau},\chi\right)\\ \left[I_2^{\langle 1\rangle}\left(\frac{2}{\tau}\right)K_2^{\langle 0\rangle}\left(\frac{2r}{\tau}\right) - \lambda^{\langle 1,0\rangle}\,\mathrm{e}^{2\xi}\left(1+\frac{\alpha^{1,0}}{2}+\frac{\beta^{1,0}}{4}+\frac{\gamma^{1,0}}{8}\right)\right]\mathrm{e}^{2\mathrm{i}\chi}\Big/ S_1^{1,0}\left(\frac{1}{\tau},\frac{r}{\tau},\chi\right)\end{matrix}\right\},$$

$$\Delta R_3(r,\tau,\chi) = 3\left\{\begin{matrix}\left[I_3^{\langle 0\rangle}\left(\frac{3r}{\tau}\right)K_3^{\langle 1\rangle}\left(\frac{3}{\tau}\right) - \lambda^{\langle 0,1\rangle}\,\mathrm{e}^{3\xi}\left(1+\frac{\alpha^{0,1}}{3}+\frac{\beta^{0,1}}{9}+\frac{\gamma^{0,1}}{27}\right)\right]\mathrm{e}^{3\mathrm{i}\chi}\Big/ S_1^{0,1}\left(\frac{r}{\tau},\frac{1}{\tau},\chi\right)\\ \left[I_3^{\langle 1\rangle}\left(\frac{3}{\tau}\right)K_3^{\langle 0\rangle}\left(\frac{3r}{\tau}\right) - \lambda^{\langle 1,0\rangle}\,\mathrm{e}^{3\xi}\left(1+\frac{\alpha^{1,0}}{3}+\frac{\beta^{1,0}}{9}+\frac{\gamma^{1,0}}{27}\right)\right]\mathrm{e}^{3\mathrm{i}\chi}\Big/ S_1^{1,0}\left(\frac{1}{\tau},\frac{r}{\tau},\chi\right)\end{matrix}\right\},$$

divided by the main part $S_M^{I,J}$ of the series in percentage. The presented data make it clear that in order to reduce the error down to 0.2% it is sufficient to add the first term ($m=1$) of remainder (3.9) to $S_M^{I,J}$

$$I_1^{\langle I\rangle}(x)K_1^{\langle J\rangle}(y)\,\mathrm{e}^{\mathrm{i}\chi} - \lambda^{\langle I,J\rangle}\,\mathrm{e}^{\xi+\mathrm{i}\chi}(1+\alpha^{I,J}+\beta^{I,J}+\gamma^{I,J}). \quad (3.10)$$

Thus, the evolution of the flow induced by helical vortex filaments reduces to calculation of the main part $S_M^{I,J}$, represented by elementary functions. Moreover, the information on vortex torsion expressly includes singularities and their coefficients in (3.8). Therefore, representation of series (2.3) by means of (3.8) even adding the remainder (3.10) is more simple and effective for the solution of the problem, rather than approaches by Ricca (1994) and Boersma & Wood (1999). Please note, that as $\tau\to\infty$, when the helical filaments become rectilinear lines, which we consider as point vortices, the coefficients $\alpha^{(i,j)}$, $\beta^{(i,j)}$, $\gamma^{(i,j)}$ tend to zero, and the solution reduces to that for the plane case, i.e. it is described by the pole for velocity field, and by the logarithm in terms for the streamfunction.

4. On the motion of multiple helical vortices

To demonstrate the efficiency of the proposed procedure for velocity field evolution via $S_M^{I,J}$, let us determine the velocity of the considered helical vortex array (figure 2), which uniformly moves along the cylinder axis with translation velocity $V=u_z(a,0)=(\Gamma N/2\pi a - a\Omega)/\tau$ and rotates with angular velocity $\Omega = u_\theta(a,0)/a$. Therefore, it is enough to define one of the motion components, e.g. Ω. Connection between translation and rotation velocities of the vortex configuration follows from (2.4) and conforms to (4.13) from Ricca (1994), but both are at variance with the erroneous trigonometric interpretation of (5.5) and (5.6) from Wood & Boersma (2001). The error was due to

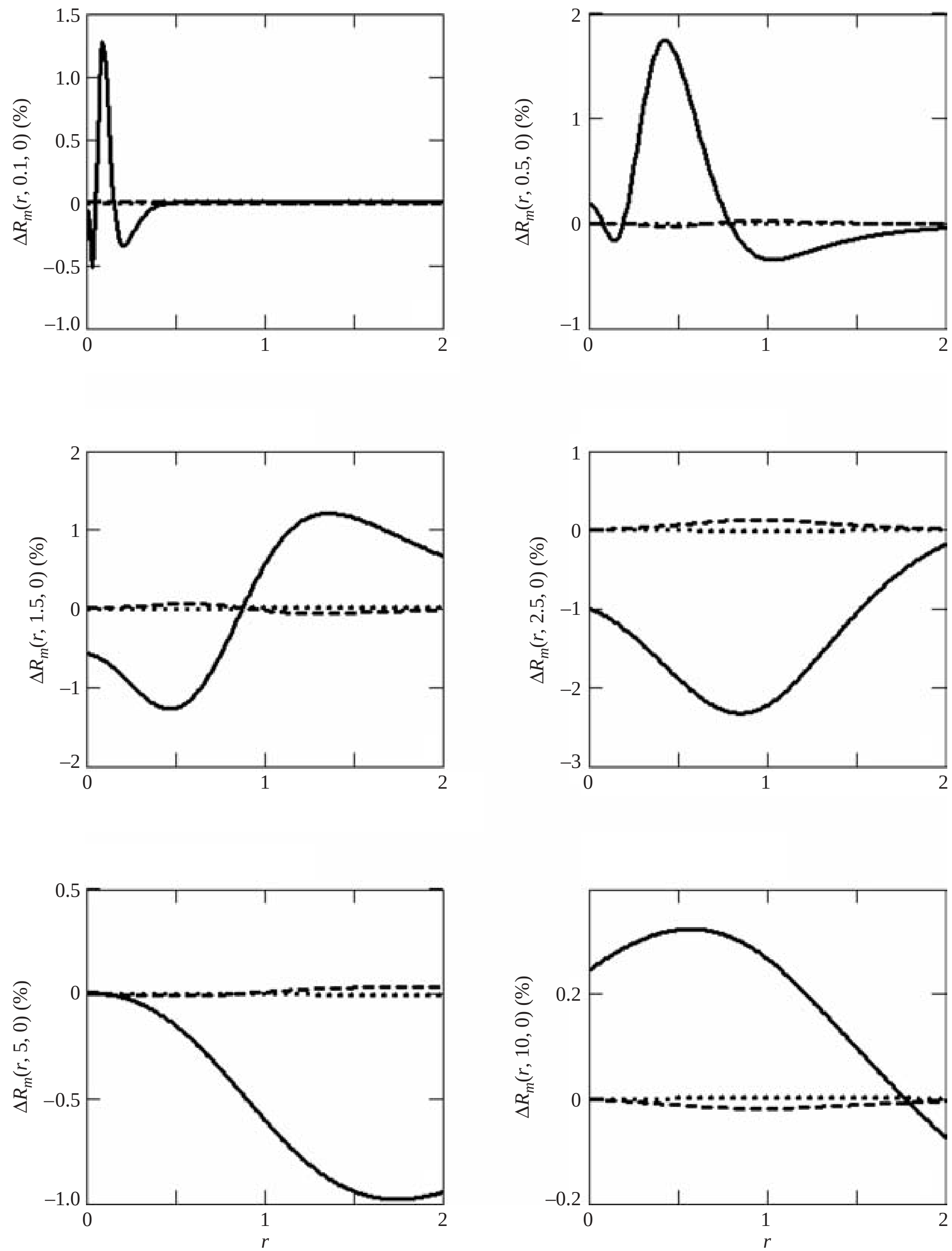

FIGURE 3. Evaluation of the regular remainder (3.9) in per cent to the main part of (3.8) for six values of helical pitch $\tau = 0.1$, 0.5, 1.5, 2.5, 5.0, 10 and for the first three terms in the series of (3.9): —, $m = 1$; ---, $m = 2$; . . . , $m = 3$.

the incorrect treatment of the helix's tangential velocity u_τ. Note that in contrast to a plane case, where vortex motion is determined by the total velocity, induced by other vortices at the point of the considered vortex, for helical vortices the total velocity consists of two components:

$$\Omega = \Omega_{Ind} + \Omega_{Sind}, \tag{4.1}$$

where Ω_{Ind} is a rotation velocity component of a fixed vortex, induced by other vortices, and Ω_{Sind} is the velocity of its self-induced motion (Saffman 1992; Ricca 1994; Kuibin & Okulov 1998; Boersma & Wood 1999). If we fix a vortex with zero angular coordinate for the considered vortex array and neglect the effect of the final core size of other vortices, substituting them by filaments, according to the superposition principle (see § 2) and formula (2.1) for the tangential velocity component, the first component of the rotation velocity (4.1) of the helical vortex array takes the form

$$\frac{4\pi a^2}{\Gamma}\Omega_{Ind} = \frac{4}{\tau}\sum_{n=1}^{N-1}\mathrm{Re}\left[\sum_{m=1}^{\infty} m I_m\left(\frac{m}{\tau}\right)K'_m\left(\frac{m}{\tau}\right)\mathrm{e}^{\mathrm{i}m(2\pi n/N)}\right] = \frac{4}{\tau}\,\mathrm{Re}\sum_{n=1}^{N-1} H_1^{0,1}\left(\frac{1}{\tau},\frac{1}{\tau},\frac{2\pi n}{N}\right). \tag{4.2}$$

In the series (4.2) we substitute $H_1^{0,1}$ by their main part $S_1^{0,1}$, adding (3.10) to improve the accuracy. After a long calculation with summation of singularities and polylogarithms placed uniformly over a circumference (Appendix D), we obtain

$$\begin{aligned}\frac{4\pi a^2}{\Gamma}\Omega_{Ind} &= N - 3 - \frac{\tau}{(1+\tau^2)^{3/2}}(\ln(N) - 1)\\ &+ \frac{\tau^3}{(1+\tau^2)^{9/2}}\left[\left(\tau^4 - 3\tau^2 + \tfrac{3}{8}\right)\left(\frac{N^2-1}{N^2}\varsigma(3) - 1\right)\right] - \frac{4}{\tau}I_1\left(\frac{1}{\tau}\right)K'_1\left(\frac{1}{\tau}\right).\end{aligned} \tag{4.3}$$

where $\zeta(\cdot)$ is the Riemann zeta function. In (4.3), we additionally used the fact that for $x = y = 1/\tau$ coefficients in (3.8) according to Appendix B, take a relatively simple form:

$$\alpha^{\langle 0,1\rangle} = -\alpha^{\langle 1,0\rangle} = \frac{1}{2}\frac{\tau}{(1+\tau^2)^{3/2}}, \quad \alpha^{\langle 0,0\rangle} = \alpha^{\langle 1,1\rangle} = 0,$$

$$\beta^{\langle 0,0\rangle} = -\beta^{\langle 1,1\rangle} = \frac{-\tau^2}{(1+\tau^2)^3}\left(\tfrac{1}{2}\tau^2 - \tfrac{1}{8}\right), \quad \beta^{\langle 0,1\rangle} = \beta^{\langle 1,0\rangle} = 0,$$

$$\gamma^{\langle 0,1\rangle} = -\gamma^{\langle 1,0\rangle} = \frac{1}{2}\frac{\tau^3}{(1+\tau^2)^{9/2}}\left(\tau^4 - 3\tau^2 + \tfrac{3}{8}\right), \quad \gamma^{\langle 0,0\rangle} = \gamma^{\langle 1,1\rangle} = 0.$$

Equation (4.3) can be applied with good approximation for all values of τ and any number of vortices in the vortex configuration. Let us prove the last statement by comparing the integral remainders W, calculated by (3.7) in Wood & Boersma (2001) and recalculated through the value of angular velocity (4.3). Thus, in accordance with (2.4), (2.7) and (3.6), note that

$$\frac{4\pi a^2}{\Gamma}\Omega_{Ind} = \frac{2(N-1)}{1+\tau^2} - \frac{\tau}{\sqrt{1+\tau^2}}u_{b_{Ind}}, \quad \frac{4\pi a^2}{\Gamma}\Omega_{Sind} = \frac{2}{1+\tau^2} - \frac{\tau}{\sqrt{1+\tau^2}}u_{b_{Sind}}, \tag{4.4}$$

where $u_{b_{Ind}}$ is a binormal velocity of a fixed vortex, induced by other vortices, $u_{b_{Sind}}$ is the velocity of its self-induced motion, and the total binormal velocity is $u_b = u_{b_{Ind}} + u_{b_{Sind}}$. After substitution of (3.6) and (4.3) into (4.4) for the cases of $N = 2$, 3 and 4 considered in Wood & Boersma (2001), and solution of the equations obtained in relation to values of $W(\pi/2, \tau)$, $W(3\pi/2, \tau)$ and $W(\pi, \tau)$, we define approximated relationships for the calculation of W in the whole variation range of τ,

$$W\left(\frac{\pi}{2},\tau\right) \approx \frac{1}{\tau} - \frac{\ln 2}{2(1+\tau^2)^{3/2}} - \frac{\tau^2}{(1+\tau^2)^{9/2}}\left(\tau^4 - 3\tau^2 + \tfrac{3}{8}\right)\frac{3}{32}\varsigma(3), \tag{4.5a}$$

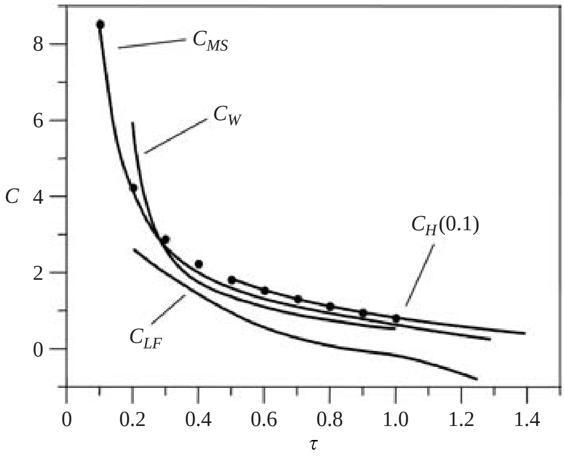

FIGURE 4. Comparison of the remainder term C for the self-induced motion of a helical vortex: C_{LF}, the asymptotic by Levy & Forsdyke (1928); C_W, Widnall (1972); C_{MS}, Ricca's asymptotic with applying the technique by Moore & Saffman (1972); C_H, Ricca's calculation based on Hardin's solution (1982). Points are the calculation based on (3.5), tabulated by Boersma & Wood (1999).

$$W\left(\frac{3\pi}{2},\tau\right)\approx\frac{2}{\tau}-\frac{\ln 3-1}{2(1+\tau^2)^{3/2}}-\frac{\tau^2(8\varsigma(3)-9)}{9(1+\tau^2)^{9/2}}\left[\left(\frac{\tau^4}{2}-\frac{3}{2}\tau^2+\frac{3}{16}\right)\right]+\frac{2}{\tau^2}I_1\left(\frac{1}{\tau}\right)K_1'\left(\frac{1}{\tau}\right), \tag{4.5b}$$

$$W(\pi,\tau)\approx\frac{3}{\tau}-\frac{\ln 2-1}{(1+\tau^2)^{3/2}}-\frac{\tau^2(3\varsigma(3)-4)}{4(1+\tau^2)^{9/2}}\left[\left(\tau^4-3\tau^2+\tfrac{3}{8}\right)\right]+\frac{4}{\tau^2}I_1\left(\frac{1}{\tau}\right)K_1'\left(\frac{1}{\tau}\right). \tag{4.5c}$$

As a result, (4.5a)–(4.5c) take a simpler form than respective asymptotics from Wood & Boersma (2001). So the proposed approach can be used for calculation of both Ω and $W$with a sufficient accuracy for any N (see comparisons between fifth and sixth, eighth and ninth, eleventh and twelfth columns of table 1).

Determination of $u_{b_{Sind}}$ or Ω_{Sind} (angular velocity of self-induced motion of every single vortex from the considered helical vortex array) is a considerably more complex problem, which has been examined by many famous workers in hydromechanics. As was mentioned in § 1, the most significant achievements in a definition of the self-induced motion were described by Ricca (1994), Kuibin & Okulov (1998) and Boersma & Wood (1999).

Let us consider the solution of the Ω_{Sind}-problem based on the approach of velocity field evolution, presented in the previous item. To this end, we will use (4.4). According to Ricca (1994), the self-induced dimensionless binormal velocity $u_{b_{Sind}}$ was defined as

$$u_{b_{Sind}}=\frac{1}{1+\tau^2}\left[\ln\left(\frac{1}{\varepsilon}\right)-\tfrac{1}{4}+C(\tau)\right]\quad\text{where } C(\tau)=\frac{\widehat{v}_{\text{int}}+\widehat{v}_{\text{ext}}}{2}-2\ln\frac{1}{\varepsilon}. \tag{4.6}$$

Thus, the solution of the problem is reduced to determination of $C(\tau)$ using formulae (3.1) and (3.2). Using these terms, Ricca (1994) compared all the previous results. His comparison of $C(\tau)$ using asymptotics by Levy & Forsdyke (1928), C_{LF}; Widnall (1972), C_W; Ricca's asymptotic, C_{MS}, derived by applying the technique of Moore & Saffman (1972), and his calculations based on the Hardin solution, C_H, are reproduced in figure 4. In addition to this, a new calculation of $C(\tau)$ carried out with the help of (3.3)–(3.4) via values of W tabulated by Boersma & Wood (1999) was indicated

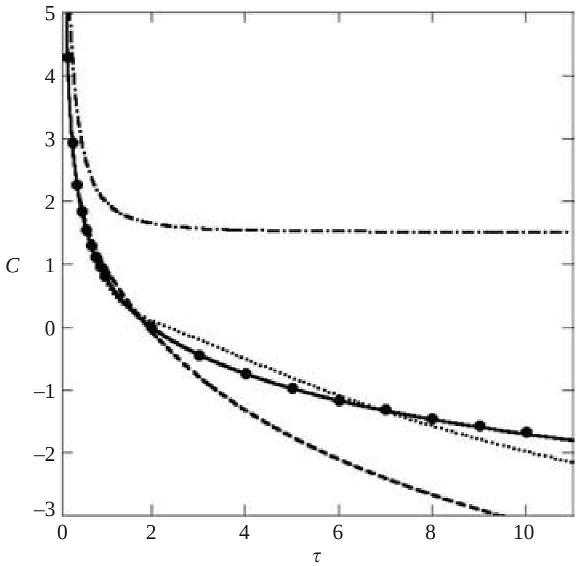

FIGURE 5. Evaluation of the remainder term C for self-induced motion of a helical vortex, using different approximating levels for the main part $S_M^{I,J}$ in (3.8): –·–, calculations using only the pole in (3.8); ---, pole and logarithm; · · ·, pole, logarithm and polylogarithms; —, coincides with the calculation according to formula (4.7) in full. Points are the calculation based on (3.5), tabulated by Boersma & Wood (1999).

by points in the same figure. Though the new $C(\tau)$ was calculated for sample points only, it is the most accurate one, and the data in figure 4, as well as in the next figure, should be considered as a standard for other asymptotics and theories. Owing to this comparison, we conclude that of the calculations by Ricca, C_H are the closest to the standard, although they were carried out numerically and for a limited number of points just as the reference calculations via data of Boersma & Wood (1999). All attempts to find an analytical form of solution (C_{LF}, C_{MS}, C_W) are not accurate enough. To construct a new analytical solution, we will use (3.1) and (3.2) replacing series $H_1^{0,1}$ and $H_1^{1,0}$ on their principal parts $S_1^{0,1}$ and $S_1^{1,0}$ with the first term of the remainder (3.10) to improve the calculation accuracy. After complex algebra, we obtain:

$$C_O(\tau) = \ln\left(\frac{\tau}{(1+\tau^2)^{3/2}}\right) - 4\frac{(1+\tau^2)^{3/2}}{\tau^2} I_1\left(\frac{1}{\tau}\right) K_1'\left(\frac{1}{\tau}\right) - \frac{\sqrt{1+\tau^2}(1-\tau\sqrt{1+\tau^2}+3\tau^2)}{\tau} + \frac{\tau^2}{(1+\tau^2)^3}\left[\left(\tau^4 - 3\tau^2 + \tfrac{3}{8}\right)\varsigma(3) - 3\tfrac{3}{8} - 2\tau^4 - \frac{1}{\tau^2}\right], \quad (4.7)$$

or using (4.6) and (4.4) we obtain for Ω_{Sind}:

$$\frac{4\pi a^2}{\Gamma}\Omega_{Sind} = 3 - \frac{\tau}{\sqrt{1+\tau^2}} - \frac{\tau}{(1+\tau^2)^{3/2}} \ln\left(\frac{\tau}{(1+\tau^2)^{3/2}}\right) + \frac{4}{\tau} I_1\left(\frac{1}{\tau}\right) K_1'\left(\frac{1}{\tau}\right) + \frac{\tau^3}{(1+\tau^2)^{9/2}}\left[3\tfrac{3}{8} + 2\tau^4 + \frac{1}{\tau^2} - \left(\tau^4 - 3\tau^2 + \tfrac{3}{8}\right)\varsigma(3)\right] - \frac{\tau}{(1+\tau^2)^{3/2}}\left(\ln\left(\frac{1}{\varepsilon}\right) - \tfrac{1}{4}\right). \quad (4.8)$$

Comparison of (4.7) in figure 5 (solid line) with the standard data (points) is in excellent agreement with the standard data. An additional test of (4.7) was made by direct comparison of Wvalues. Using (3.1)–(3.4) and (4.5)–(4.7) in (4.3), we find the approximation for $W(\tau)$:

$$W(\tau) \approx \frac{1}{\sqrt{1+\tau^2}} - \frac{1}{\tau} + \frac{1}{(1+\tau^2)^{3/2}} \ln\left(\frac{\tau}{2(1+\tau^2)} - 2\tau^2\right) - \frac{4}{\tau^2} I_1\left(\frac{1}{\tau}\right) K_1'\left(\frac{1}{\tau}\right) + \frac{\tau^2}{(1+\tau^2)^9}\left[\left(\tau^4 - 3\tau^2 + \tfrac{3}{8}\right)\varsigma(3) - 3\tfrac{3}{8} - 2\tau^4 - \frac{1}{\tau^2}\right], \quad (4.9)$$

in the whole variation range of τ (see comparison between the second and third columns in table 1).

In figure 5, it is also shown how the solution may differ if the main part of (3.8) takes into account pole solely; pole and logarithm; pole plus logarithm, and polylogarithms without the first term of the remainder (3.10). Note that good comparison can be obtained only if all components of (3.8) and the first term of (3.10) of the remainder part of the series (2.3) are considered. Thus, it is impossible to reach the required accuracy in calculation of angular velocity of self-induced motion Ω_{Sind} and velocity induced by other velocity vortices Ω_{Ind}, if they are presented by elementary algebraic functions only. In (4.3) and (4.8), just as in approximations (4.5) and (4.9), we must consider the component presented via the product of special functions (modified Bessel functions). However, in presentation of the total angular velocity $\Omega = \Omega_{Ind} + \Omega_{Sind}$, these components disappear. The formula for the angular moving velocity of N helical vortex arrays takes the simple algebraic form:

$$\frac{4\pi a^2}{\Gamma}\Omega = \frac{4\pi a^2}{\Gamma}(\Omega_{Ind} + \Omega_{Sind}) = N - \frac{\tau}{\sqrt{1+\tau^2}} - \frac{\tau}{(1+\tau^2)^{3/2}}\left(\ln\left(\frac{\tau}{N(1+\tau^2)^{3/2}}\right) + 1\right) + \frac{\tau^3}{(1+\tau^2)^{9/2}}\left[\tau^4 + 3\tau^2 + 3 + \frac{1}{\tau^2} - \left(\tau^4 - 3\tau^2 + \tfrac{3}{8}\right)\frac{\varsigma(3)}{N^2}\right] - \frac{\tau}{(1+\tau^2)^{3/2}}\left(\ln\left(\frac{1}{\varepsilon}\right) - \tfrac{1}{4}\right). \quad (4.10)$$

Note that according to (4.4) for total velocities and (5.7) from Wood & Boersma (2001), the angular rotation velocity of a helical vortex array can be written using W:

$$\frac{4\pi a^2}{\Gamma}\Omega_{WB} = \frac{2N}{1+\tau^2} - \frac{\tau}{\sqrt{1+\tau^2}} u_b = 2N - \tau\left\{W(\tau) + \sum_{n=1}^{N-1} W\left(\frac{2\pi n}{N}, \tau\right)\right\} - \frac{\tau}{(1+\tau^2)^{3/2}}\left(\ln 2 + 2\tau^2 - \tfrac{1}{2}\ln(1+\tau^2)\right) - \frac{\tau}{(1+\tau^2)^{3/2}}\left(\ln\left(\frac{1}{\varepsilon}\right) - \tfrac{1}{4}\right). \quad (4.11)$$

Equation (4.11) allows us to test our result (4.10). In table 2, angular velocities Ω' calculated without the last component in formula (4.10) are compared with the values of Ω'_{WB} calculated also without the last identical components in (4.11). This comparison was made for 21 values of τ, for which integrals (3.5) and (3.7) were calculated in Boersma & Wood (1999) and Wood & Boersma (2001). Note that for the whole range of dimensionless pitch τ, maximal difference does not exceed 0.4% for $N = 2$ and the difference decreases with growth of N. In addition, we emphasize that (4.10) takes a simpler form as compared with the asymptotic forms from Wood & Boersma (2001).

τ	$N = 2$ (4.11) by WB	$N = 2$ (4.10) present data	$N = 2$ Difference (%)	$N = 3$ (4.11) by WB	$N = 3$ (4.10) present data	$N = 3$ Difference (%)	$N = 4$ (4.11) by WB	$N = 4$ (4.10) present data	$N = 4$ Difference (%)
0.01	2.04298	2.04298	0.0	3.04703	3.04703	0.0	4.04991	4.04991	0.0
0.05	2.13399	2.13399	0.0	3.15419	3.15419	0.0	4.16853	4.16853	0.0
0.1	2.19700	2.19700	0.0	3.23700	3.23700	0.0	4.26536	4.26536	0.0
0.2	2.24867	2.24867	0.0	3.32542	3.32541	0.0	4.37976	4.37976	0.0
0.3	2.24632	2.24623	−0.009	3.35348	3.35346	−0.002	4.42943	4.42943	0.0
0.4	2.21636	2.21596	−0.041	3.34541	3.34534	−0.008	4.43731	4.43729	−0.002
0.5	2.17347	2.17281	−0.065	3.31564	3.31549	−0.016	4.41763	4.41758	−0.005
0.6	2.12516	2.12482	−0.034	3.27312	3.27301	−0.011	4.38006	4.38002	−0.004
0.7	2.07484	2.07545	0.061	3.22329	3.22337	0.008	4.33122	4.33124	0.002
0.8	2.02398	2.02584	0.186	3.16943	3.16981	0.038	4.27561	4.27573	0.012
0.9	1.97337	1.97632	0.295	3.11372	3.11438	0.066	4.21636	4.21658	0.022
1	1.92357	1.92714	0.357	3.05766	3.05850	0.084	4.15573	4.15601	0.029
2	1.53243	1.52990	−0.253	2.60726	2.60662	−0.064	3.65977	3.65954	−0.023
3	1.32902	1.32656	−0.246	2.37333	2.37278	−0.055	3.40296	3.40278	−0.018
4	1.22221	1.22119	−0.102	2.25097	2.25079	−0.018	3.26953	3.26947	−0.006
5	1.16050	1.16031	−0.019	2.18053	2.18053	0.000	3.19310	3.19310	0.001
6	1.12175	1.12195	0.019	2.13645	2.13652	0.007	3.14547	3.14550	0.003
7	1.09582	1.09617	0.035	2.10703	2.10712	0.009	3.11381	3.11384	0.003
8	1.07757	1.07797	0.040	2.08638	2.08648	0.010	3.09165	3.09169	0.004
9	1.06421	1.06462	0.041	2.07131	2.07142	0.010	3.07553	3.07556	0.003
10	1.05413	1.05452	0.039	2.05998	2.06006	0.009	3.06341	3.06344	0.003

TABLE 2. Comparison of angular velocities $4\pi a^2 \Omega'/\Gamma$ with the values of $4\pi a^2 \Omega'_{WB}/\Gamma$ for different numbers of helical vortices in the arrays and several values of helical pitch τ fixed in Wood & Boersma (2001). The results for (4.10) and (4.11) were calculated without the last identical components.

5. Stability of the multiple helical vortices

Clearly, there is an equilibrium in which the multiple helical vortices uniformly move along the cylinder axis with translation velocity $V = (\Gamma N/2\pi a - a\Omega)/\tau$ and rotate with angular velocity Ω. In order to formulate the problem on linear stability of the equilibrium motion for multiple helical vortices, we introduced in § 2 the helical variables (r, χ) and the corresponding velocity components $(u_r, u_\chi = u_\theta - u_z/\tau)$. In this case, the problem reduces to a two-dimensional task (Okulov 2002), which can be solved according to the solution of a classical problem on linear stability of equilibrium for a polygonal array of point vortices (Saffman 1992).

Let the kth vortex ($k \in [0, N-1]$) be displaced from the equilibrium position to the point $(a + r_k, 2\pi k/N + t(u_\chi/a) + \chi_k)$ or $(a + r_k, 2\pi k/N + (\Omega - V/l)t + \chi_k)$, then the equations of motion of the kth disturbed vortex can be written as:

$$\frac{\mathrm{d}}{\mathrm{d}t}(a + r_k) = \sum_{n(n\neq k)} u_r\left(a + r_k, a + r_n, \frac{2\pi(n-k)}{N} + \chi_k - \chi_n\right), \tag{5.1a}$$

$$(a + r_k)\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{2\pi k}{N} + (\Omega - V/l)t + \chi_k\right) = \sum_{n(n\neq k)} u_\chi\left(a + r_k, a + r_n, \frac{2\pi(n-k)}{N} + \chi_k - \chi_n\right). \tag{5.1b}$$

The velocities are given by (2.1) and (2.5). Linearizing in r_k, r_n, χ_k, χ_n and substituting the velocity field derivatives (see Appendix A), a linear approximation of (5.1) for motion of the kth vortex can be written after some algebra as:

$$\frac{\mathrm{d}r_k}{\mathrm{d}t} = \frac{\Gamma}{\pi a}\frac{1}{\tau^2}\,\mathrm{Re}\left\{\chi_k \sum_{n(n\neq k)} H_2^{1,1}\left(a, a, \frac{2\pi(n-k)}{N}\right) - \sum_{n(n\neq k)} \chi_n H_2^{1,1}\left(a, a, \frac{2\pi(n-k)}{N}\right)\right\}, \tag{5.2a}$$

$$\begin{aligned} a\frac{\mathrm{d}\chi_k}{\mathrm{d}t} = \frac{\Gamma}{\pi a^2}\frac{1}{\tau^2}\mathrm{Re}\Bigg\{ r_k \sum_{n(n\neq k)} \Bigg[\frac{\tau^2+1}{\tau^2} H_2^{1,1}\left(a, a, \frac{2\pi(n-k)}{N}\right) \\ + \frac{1-\tau^2}{\tau} H_1^{0,1}\left(a, a, \frac{2\pi(n-k)}{N}\right)\Bigg] + \frac{(\tau^2+1)^2}{\tau^2}\sum_{n(n\neq k)} r_n H_2^{1,1}\left(a, a, \frac{2\pi(n-k)}{N}\right)\Bigg\} \\ - \left(\frac{\tau^2+1}{\tau^2}\Omega - \frac{\Gamma N}{2\pi a^2\tau^2}\right) r_k. \end{aligned} \tag{5.2b}$$

Following Saffman (1992), we will look for the solution of system (5.2) in the following form:

$$r_k = \alpha(t)\,\mathrm{e}^{2\pi k m/N}, \qquad \chi_k = \beta(t)\,\mathrm{e}^{2\pi k m/N}.$$

Here, m is the subharmonic wavenumber, taking integral values within the range $[0, N-1]$. For $m = 0$, all vortices behave in the same way.

As a result, we obtain the following system of equations:

$$\begin{aligned} \alpha'(t) &= A(m)\beta(t), \\ \beta'(t) &= B(m)\alpha(t), \end{aligned}$$

from which it follows that α and β are proportional to $\exp(t\sqrt{AB})$. Therefore, if there is any m, for which $AB \geqslant 0$, then the system is unstable. Substituting $H_M^{I,J}$ for $S_M^{I,J}$ by (3.8), and after time-consuming algebra, we were able to find an algebraic form for product AB:

$$\frac{16\pi^2 a^3}{\Gamma^2}AB = \left[m(N-m)\frac{\sqrt{1+\tau^2}}{\tau} - \frac{\tau}{4}\frac{4\tau^2-3}{(1+\tau^2)^{5/2}}\left(\frac{N}{m} - C - \psi\left(-\frac{m}{N}\right)\right)\right]$$
$$\times\left[m(N-m)\frac{\sqrt{1+\tau^2}}{\tau} - N + \frac{\tau}{\sqrt{1+\tau^2}} + (N-1)\frac{3-\tau^2}{1+\tau^2}\right.$$
$$+\frac{\tau}{(1+\tau^2)^{3/2}}\left(\ln\left(\frac{\tau}{\sigma N\sqrt{1+\tau^2}}\right) + \tfrac{3}{4}\right)$$
$$+\frac{\tau^3}{(1+\tau^2)^{5/2}}\left(\ln(N) - \left(1 - \frac{1}{4\tau^2}\right)\left(\frac{N}{m} - C - \psi\left(-\frac{m}{N}\right)\right)\right)$$
$$+\frac{\tau^3}{(1+\tau^2)^{9/2}}\left[\left(\tau^4 - 3\tau^2 + \tfrac{3}{8}\right)\frac{\varsigma(3)}{N^2} - \tau^4 - 3\tau^2 - 3 - \frac{1}{\tau^2}\right]$$
$$\left.+\frac{\tau^3(1-\tau^2)}{(1+\tau^2)^{11/2}}\left(\tau^4 - 3\tau^2 + \tfrac{3}{8}\right)\frac{N^2-1}{N^2}\varsigma(3)\right], \tag{5.3}$$

where $C = 0.577215\ldots$ is the Euler constant, and $\psi(\cdot)$ is the psi function. Note that (5.3) at $\tau \to \infty$ reduced to the formula for the case of point vortices (Saffman 1992),

$$\frac{16\pi^2 a^3}{\Gamma^2}AB = m(N-m)[m(N-m) - 2(N-1)].$$

Analysing (5.3) at various values of pitch τ, we determined unstable modes (figure 6), which are more realistic for analysing for the existence of equilibrium for real N-vortex arrays in a tornado, after vortex breakdown, in vortex chambers etc. rather than the solution for the point vortex array (Saffman 1992). In order to compare the arrays with different numbers of vortices to keep total intensity in a system, the vortex size was varied so that the total area of core cross-sections was constant, i.e. $\sigma_N = 0.4/\sqrt{N}$. Note, that decreasing the vortex pitch leads to loss of stability for fewer and fewer vortices in the arrays, and at $\tau \leqslant 1.5$, the stable helical vortex arrays are completely absent for the fixed vortex core. Qualitatively, these data agree with the results of visual observations of the multiple helical vortices without hub vortex. Besides, in order to evaluate the influence of vortex core sizes, neutral curves for several equilibrium circular helical vortex arrays were plotted in figure 7. Experimental results of Alekseenko *et al.* (1999) allow us to carry out quantitative comparison as well. Alekseenko *et al.* (1999) describe the double vortex structure ($N = 2$) with dimensionless pitch $\tau = 1.5$. Though this pattern is close to the instability boundary (see figure 7), it is still stable, i.e. such a vortex couple can exist. Proximity of the vortex parameters to the boundary of unstable regimes is confirmed indirectly by the fact that it was quite difficult to obtain such vortices during the experiment. Very fine adjustment of the experimental set-up, as well as the flow regimes, was required to obtain such a vortex couple.

6. Conclusion

In the present work, a classical problem of stability of a circular equilibrium array of N point vortices to infinitesimal space displacements is, for the first time, generalized for the case of multiple N helical vortex arrays. Consequently, the analytical solution has been obtained. This solution allows us to provide an efficient analysis of the experimentally observed stable vortex arrays (couples, triplets, etc.)

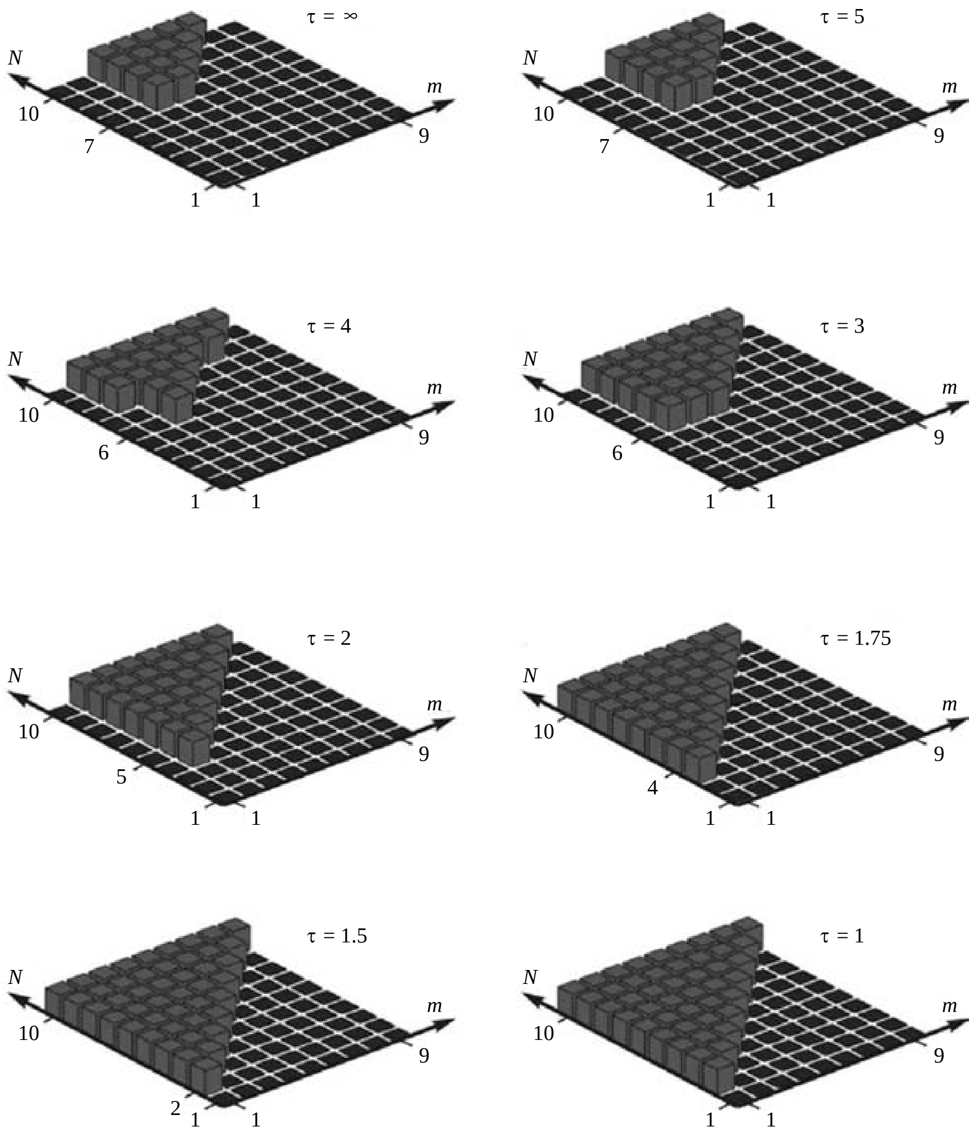

FIGURE 6. Diagrams of unstable subharmonic wavenumbers (m) for the circular equilibrium array of N helical vortices with conservation of a total cross-section area for vortex cores, i.e. $\sigma_N = 0.4/\sqrt{N}$.

The algebraic presentation allowing us to carry out high-accuracy calculations within the whole range of helical pitch alteration is obtained as an additional intermediate result for the angular rotation velocity of the N helical vortex array. The new formula has a simpler form as compared with the known asymptotics (Boersma & Wood 1999; Wood & Boersma 2001).

Solution of the above two classical problems of vortex dynamics became possible because of the discrimination in the velocity field of the principal part induced by an infinitely slender vortex filament. The principal part was expressed in the form of the sum of elementary algebraic functions: pole plus logarithm with a small regular

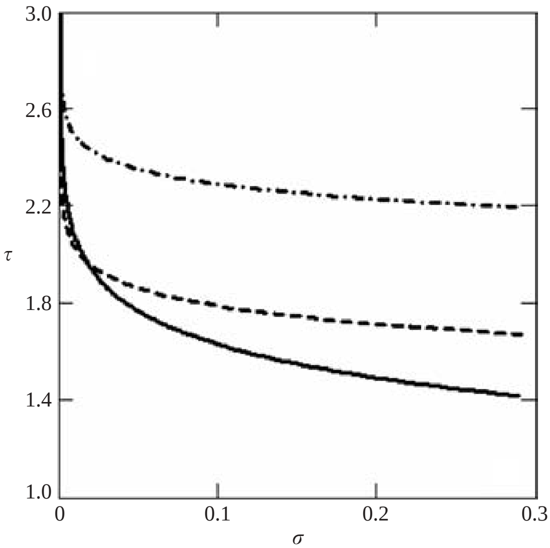

FIGURE 7. Neutral stability curves for the circular equilibrium arrays of helical vortices: —, $N=2$; ---, $N=3$; –·–, $N=4$ (stable region above each curve).

remainder. Such an approach and the solution of the problems are important both for theoretic and applied hydromechanics, as well as for many other areas of natural science, where a rotor (vortex) concept is the basic idea.

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Appendix A. Derivatives of velocity field induced by helical vortex filament

Here are some forms of velocity field $\{u_r, u_{\chi},\}$ derivatives, induced in unbounded space by infinitely slender helical vortex filament:

$$\frac{\partial u_r}{\partial r} = \frac{\Gamma a}{\pi r^2 l}\frac{r^2+l^2}{l^2}\,\mathrm{Im}\left\{\begin{matrix} H_2^{0,1}(r/l, a/l, \chi-\chi_0) \\ H_2^{1,0}(a/l, r/l, \chi-\chi_0)\end{matrix}\right\} - \frac{\Gamma a}{\pi r l^2}\,\mathrm{Im}\left\{\begin{matrix} H_1^{1,1}(r/l, a/l, \chi-\chi_0) \\ H_1^{1,1}(a/l, r/l, \chi-\chi_0)\end{matrix}\right\},$$

$$\frac{\partial u_r}{\partial a} = \frac{\Gamma}{\pi a l}\frac{r^2+l^2}{l^2}\,\mathrm{Im}\left\{\begin{matrix} H_2^{0,1}(r/l, a/l, \chi-\chi_0) \\ H_2^{1,0}(a/l, r/l, \chi-\chi_0)\end{matrix}\right\},$$

$$\frac{\partial u_r}{\partial \chi} = -\frac{\Gamma a}{\pi l^2}\,\mathrm{Re}\left\{\begin{matrix} H_2^{1,1}(r/l, a/l, \chi-\chi_0) \\ H_2^{1,1}(a/l, r/l, \chi-\chi_0)\end{matrix}\right\},$$

$$\frac{\partial u_r}{\partial \chi_0} = \frac{\Gamma a}{\pi l^2}\,\mathrm{Re}\left\{\begin{matrix} H_2^{1,1}(r/l, a/l, \chi-\chi_0) \\ H_2^{1,1}(a/l, r/l, \chi-\chi_0)\end{matrix}\right\},$$

$$\frac{\partial u_{\chi}}{\partial r} = -\frac{\Gamma}{\pi l^2} + \frac{\Gamma a}{\pi r l^2}\frac{r^2+l^2}{l^2}\,\mathrm{Re}\left\{\begin{matrix} H_2^{1,1}(r/l, a/l, \chi-\chi_0) \\ H_2^{1,1}(a/l, r/l, \chi-\chi_0)\end{matrix}\right\}$$
$$-\frac{\Gamma a}{\pi r^2 l}\frac{r^2-l^2}{l^2}\,\mathrm{Re}\left\{\begin{matrix} H_1^{0,1}(r/l, a/l, \chi-\chi_0) \\ H_1^{1,0}(a/l, r/l, \chi-\chi_0)\end{matrix}\right\},$$

$$\frac{\partial u_\chi}{\partial a} = \frac{\Gamma}{\pi a r}\frac{r^2+l^2}{l^2}\frac{a^2+l^2}{l^2}\mathrm{Re}\left\{\begin{matrix} H_2^{0,0}(r/l, a/l, \chi-\chi_0) \\ H_2^{0,0}(a/l, r/l, \chi-\chi_0)\end{matrix}\right\},$$

$$\frac{\partial u_\chi}{\partial \chi} = -\frac{\Gamma a}{\pi r l}\frac{r^2+l^2}{l^2}\mathrm{Im}\left\{\begin{matrix} H_2^{0,1}(r/l, a/l, \chi-\chi_0) \\ H_2^{1,0}(a/l, r/l, \chi-\chi_0)\end{matrix}\right\},$$

$$\frac{\partial u_\chi}{\partial \chi_0} = \frac{\Gamma a}{\pi r l}\frac{r^2+l^2}{l^2}\mathrm{Im}\left\{\begin{matrix} H_2^{0,1}(r/l, a/l, \chi-\chi_0) \\ H_2^{1,0}(a/l, r/l, \chi-\chi_0)\end{matrix}\right\}.$$

The derivatives are obtained beyond the filament points by direct differentiation of Hardin's solution (Hardin 1982) employing recurrent relations between the modified cylindrical functions (Abramovitz & Stegun 1964).

Appendix B. Asymptotic presentations of cylindrical functions

To complete the statement of § 3, we present here the asymptotic expansion of modified cylindrical functions at high orders according to formulae (9.7.7)–(9.7.10) from Abramovitz & Stegun (1964):

$$I_m(mx) = \frac{\mathrm{e}^{m\eta}\sqrt{t}}{\sqrt{2\pi m}}\left[1+\frac{\vartheta_1}{m}+\frac{\vartheta_2}{m^2}+\frac{\vartheta_3}{m^3}+\frac{\vartheta_4}{m^4}\dots\right],$$

$$I'_m(mx) = \frac{\mathrm{e}^{m\eta}}{x\sqrt{2\pi m t}}\left[1+\frac{\upsilon_1}{m}+\frac{\upsilon_2}{m^2}+\frac{\upsilon_3}{m^3}+\frac{\upsilon_4}{m^4}\dots\right],$$

$$K_m(mx) = \sqrt{\frac{\pi t}{2m}}\,\mathrm{e}^{-m\eta}\left[1-\frac{\vartheta_1}{m}+\frac{\vartheta_2}{m^2}-\frac{\vartheta_3}{m^3}+\frac{\vartheta_4}{m^4}\dots\right],$$

$$K'_m(mx) = -\frac{\mathrm{e}^{-m\eta}\sqrt{\pi}}{x\sqrt{2mt}}\left[1-\frac{\upsilon_1}{m}+\frac{\upsilon_2}{m^2}-\frac{\upsilon_3}{m^3}+\frac{\upsilon_4}{m^4}\dots\right],$$

$$\vartheta_1 = (3t-5t^3)/24, \quad \upsilon_1 = (-9t+7t^3)/24,$$

$$\vartheta_2 = (81t^2-462t^4+385^6)/1152, \quad \upsilon_2 = (-135t^2+594t^4-455t^6)/1152,$$

$$\vartheta_3 = (30375t^3-369603t^5+765765t^7-425425t^9)/414720,$$

$$\upsilon_3 = (-42525t^3+451737t^5-883575t^7+475475t^9)/414720,$$

$$\vartheta_4 = (4465125t^4-94121676t^6+349922430t^8-446185740t^{10}+185910725t^{12})/39813120,$$

$$\upsilon_4 = (-5740875t^4+111234708t^6-396578754t^8+493152660t^{10} \\ -202076875t^{12})/39813120,$$

where

$$\eta = \frac{1}{t}+\tfrac{1}{2}\ln\frac{1-t}{1+t}, \quad t=(1+x^2)^{-1/2}, \quad x \text{ or } y = \frac{a}{l} \text{ or } \frac{r}{l}.$$

Note the form of coefficients $\alpha^{i,j}$, $\beta^{i,j}$, $\gamma^{i,j}$ from (3.8), which are obtained as a result of multiplication of the above written uniform expansions:

$$\alpha^{i,j}(x,y) = (1-i)\vartheta_1(x)-(1-j)\vartheta_1(y)+i\upsilon_1(x)-j\upsilon_1(y),$$

$$\beta^{i,j}(x,y) = (1-i)\vartheta_2(x)+(1-j)\vartheta_2(y)+i\upsilon_2(x)+j\upsilon_2(y)-(1-i)(1-j)\vartheta_1(x)\vartheta_1(y) \\ -j(1-i)\,\vartheta_1(x)\upsilon_1(y)-i(1-j)\vartheta_1(y)\upsilon_1(x)-ij\upsilon_1(x)\upsilon_1(y),$$

$$\gamma^{i,j}(x,y) = (1-i)\vartheta_3(x) - (1-j)\vartheta_3(y) + i\upsilon_3(x) - j\upsilon_3(y) + (1-i)(1-j)[\vartheta_1(x)\vartheta_2(y) - \vartheta_2(x)\vartheta_1(y)] + j(1-i)[\vartheta_1(x)\upsilon_2(y) - \vartheta_2(x)\upsilon_1(y)] + i(1-j)[\vartheta_2(y)\upsilon_1(x) - \vartheta_1(y)\upsilon_2(x)] + ij[\upsilon_1(x)\upsilon_2(y) - \upsilon_2(x)\upsilon_1(y)].$$

Appendix C. Series development of poles, logarithms and polylogarithms

For convenience in reading the paper, we present here some expansions for the elementary functions, which can be found in the readily available mathematical reference books, such as Prudnikov, Brychkov & Marichev (1981, 1986).

$$\frac{\exp^{(\xi+\mathrm{i}\chi)}}{\left(\exp^{(\xi)} - \exp^{(\mathrm{i}\chi)}\right)^2} = \sum_{m=1}^{\infty} m \exp[-m(\xi - \mathrm{i}\chi)],$$

$$\frac{\exp^{(\mathrm{i}\chi)}}{\exp^{(\xi)} - \exp^{(\mathrm{i}\chi)}} = \sum_{m=1}^{\infty} \exp[-m(\xi - \mathrm{i}\chi)],$$

$$\ln\left(1 - \exp^{(-\xi+\mathrm{i}\chi)}\right) = \sum_{m=1}^{\infty} \frac{1}{m} \exp[-m(\xi - \mathrm{i}\chi)],$$

$$\mathrm{Li}_2\left(\exp^{(-\xi+\mathrm{i}\chi)}\right) = \sum_{m=1}^{\infty} \frac{1}{m^2} \exp[-m(\xi - \mathrm{i}\chi)],$$

$$\mathrm{Li}_3\left(\exp^{(-\xi+\mathrm{i}\chi)}\right) = \sum_{m=1}^{\infty} \frac{1}{m^3} \exp[-m(\xi - \mathrm{i}\chi)].$$

Appendix D. Sum of the poles, logarithms and polylogarithms, situated on the circumference with identical displacement

Here are several sums of peculiarities and polylogarithms in the points, situated on the circumference with identical displacement:

$$\sum_{n=1}^{N-1} \frac{\exp(\mathrm{i}2\pi n/N)}{[1 - \exp(\mathrm{i}2\pi n/N)]^2} = \frac{1-N^2}{12}, \quad \sum_{n=1}^{N-1} \frac{\exp[\mathrm{i}2\pi n(1+m)/N]}{[1 - \exp(\mathrm{i}2\pi n/N)]^2} = \frac{1-N^2}{12} - \frac{m(N-m)}{2},$$

$$\sum_{n=1}^{N-1} \frac{\exp(\mathrm{i}2\pi n/N)}{1 - \exp(\mathrm{i}2\pi n/N)} = \frac{1-N}{2}, \quad \sum_{n=1}^{N-1} \frac{\exp[\mathrm{i}2\pi n(1+m)/N]}{1 - \exp(\mathrm{i}2\pi n/N)} = \frac{1-N}{2} + m,$$

$$\sum_{n=1}^{N-1} \ln\left[1 - \exp\left(\mathrm{i}\frac{2\pi n}{N}\right)\right] = \ln(N),$$

$$\sum_{n=1}^{N-1} \exp\left(\mathrm{i}\frac{2\pi nm}{N}\right) \ln\left[1 - \exp\left(\mathrm{i}\frac{2\pi n}{N}\right)\right] = \ln(N) + E + \psi\left(\frac{-m}{N}\right) - \frac{N}{m},$$

$$\sum_{n=1}^{N-1} \mathrm{Li}_2\left(\exp\left(\mathrm{i}\frac{2\pi n}{N}\right)\right) = \frac{\pi^2}{6}\frac{1-N}{N},$$

$$\sum_{n=1}^{N-1} \exp\left(\mathrm{i}\frac{2\pi nm}{N}\right) \mathrm{Li}_2\left(\exp\left(\mathrm{i}\frac{2\pi n}{N}\right)\right) = \frac{1}{N}\varsigma\left(2, \frac{m}{N}\right) - \frac{\pi^2}{6},$$

$$\sum_{n=1}^{N} \mathrm{Li}_2\left(\exp\left(\mathrm{i}\frac{2\pi n}{N}\right)\right) = \frac{\pi^2}{6}\frac{1}{N}, \quad \sum_{n=1}^{N} \exp\left(\mathrm{i}\frac{2\pi nm}{N}\right) \mathrm{Li}_2\left(\exp\left(\mathrm{i}\frac{2\pi n}{N}\right)\right) = \frac{1}{N}\varsigma\left(2, \frac{m}{N}\right),$$

$$\sum_{n=1}^{N-1} \mathrm{Li}_3\left(\exp\left(\mathrm{i}\frac{2\pi n}{N}\right)\right) = \frac{1-N^2}{N^2}\varsigma(3),$$

$$\sum_{n=1}^{N-1} \exp\left(\mathrm{i}\frac{2\pi nm}{N}\right) \mathrm{Li}_3\left(\exp\left(\mathrm{i}\frac{2\pi n}{N}\right)\right) = \frac{1}{N^2}\varsigma\left(3, \frac{m}{N}\right) - \varsigma(3),$$

$$\sum_{n=1}^{N} \mathrm{Li}_3\left(\exp\left(\mathrm{i}\frac{2\pi n}{N}\right)\right) = \frac{1}{N^2}\varsigma(3), \quad \sum_{n=1}^{N} \exp\left(\mathrm{i}\frac{2\pi nm}{N}\right) \mathrm{Li}_3\left(\exp\left(\mathrm{i}\frac{2\pi n}{N}\right)\right) = \frac{1}{N^2}\varsigma\left(3, \frac{m}{N}\right),$$

$$\sum_{n=1}^{N-1} \exp\left(\mathrm{i}\frac{2\pi nm}{N}\right) = \begin{cases} N-1 & \text{if} \quad m = N, \\ -1 & \text{if} \quad m \neq N. \end{cases}$$

The first four sums were used to study the problem of stability of polygonal arrays of point vortices (Saffman 1992), the fifth and sixth sums can be found in, for example, Prudnikov *et al.* (1981, 1986), while the sums with polylogarithms have been obtained by the author independently.

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